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1999 J. Phys. A: Math. Gen. 32 1475

(http://iopscience.iop.org/0305-4470/32/8/014)

J. Phys. A: Math. Gen. **32** (1999) 1475–1486. Printed in the UK PII: S0305-4470(99)90887-9

The non-dynamical r-matrix structure for the elliptic A_{n-1} Calogero–Moser model

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Received 20 January 1998, in final form 23 November 1998

Abstract. In this paper we construct a new Lax operator for the elliptic A_{n-1} Calogero–Moser model with general n ($n \geqslant 2$) from the classical dynamical twisting, in which the corresponding r-matrix is purely numerical (non-dynamic). The non-dynamical r-matrix structure of this Lax operator is obtained, which is the elliptic Z_n-symmetric r-matrix.

1. Introduction

A general description of classical completely integrable models of n one-dimensional particles with two-body interactions $V(q_i - q_j)$ was given in [24]. With each simple Lie algebra and choice of one of these types of interactions one can associate a classically completely integrable systems [5, 13, 23, 24]. The most general form of the potential in such models is the so-called elliptic Calogero–Moser (CM) model with an elliptic interaction potential. The various degeneracies of this general system yield the rational CM model (type I in [5]), the hyperbolic CM model (type II in [5]) and the trigonometric CM model (type III in [5]). So, the study of the elliptic CM model is of great importance in completely integrable particle systems.

The Lax pair representation (Lax representation) of a completely integrable system, which means that the equations in the problem can be formulated in a Lax form, is the most effective way of demonstrating its integrability and constructing the complete set of integrals of motion. The Lax representation and its corresponding r-matrix structure for rational, hyperbolic and trigonometric A_{n-1} CM models were constructed by Avan *et al* [5]; the Lax representation for the elliptic CM models was constructed by Krichever [22] and the corresponding r-matrix structure was given by Sklyanin [30] and Braden *et al* [10]. A specific feature exists in that the r-matrix of the Lax representation for these models turns out to be a dynamical one (i.e. it depends upon the dynamical variables) and satisfies a dynamical Yang–Baxter equation [6, 10, 11, 30]. Such structures also appear in the study of Ruijsenaars–Schneider models, which are known as the relativistic Calogero–Moser models and can be related to the soliton systems of the affine Toda field theories [11, 12, 27, 28]. Moreover, such a dynamical r-matrix structure is connected with the Hamiltonian reduction of the cotangent bundle of the Lie algebra for the Calogero–Moser model and the contangent bundle of the Lie group for the Ruijsenaars–Schneider model [2, 16, 31]. This greatly promotes the study of the classical (and quantum) dynamical r-matrices (and R-matrix). A partial classification scheme

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0305-4470/99/081475+12$19.50

has very recently been proposed for the dynamical r-matrix obeying the particular version of the dynamical Yang–Baxter equation [3, 14, 29]. However, at the time of writing a general classification scheme such as exists in the case of non-dynamical classical r-matrices thanks to Belavin and Drinfeld [9] is still lacking.

Other difficulties presented by the dynamical aspect of the r-matrix also occur. (i) The fundamental Poisson algebra of the Lax operator, whose structural constants are given by a dynamical r-matrix, is generally speaking no longer closed (cf the non-dynamical one). (ii) Solution of the quantization problem and its geometrical interpretation is still an open problem [30]. So far, only for one particular case—the spin generalization of the CM model—has a proper algebraic setting (the Gervais–Neveu–Felder equation) been found [4] which allows one to quantize the model. On the other hand, it is well known that the Lax representation for a completely integrable models is not unique. The different Lax representations of an integrable system are transformed under a similarity transformation from each other (only for a finite-particle system, but for the field system it should be transformed under gauge transformation from each other). However, the corresponding r-matrix should be transformed under a 'gauge' transformation (see equation (5)), which is the classical dynamically twisting relations [4] between the r-matrix. So, to overcome the above difficulties caused by the dynamical r-matrix, the question arises as to whether another Lax representation for the CM model which has a non-dynamical r-matrix structure can be found. The plan of our work is to find such a 'good' Lax representation for the elliptic A_{n-1} CM model if it exists. In a previous article [18], we succeeded in constructing a new Lax operator (cf Krichever's [22]) for the elliptic A_{n-1} CM model with $n = 2$ and showing that the corresponding r-matrix is a non-dynamical one, this being the classical eight-vertex r-matrix [20]. In the present paper, extending our previous results [18], we construct a new Lax operator (cf Krichever's) for the elliptic A_{n-1} CM model with general n ($n \geqslant 2$) which is 'good' in sense that the corresponding r-matrix is non-dynamical, namely the classical Z_n-symmetric r-matrix.

The paper is organized as follows. In section 2, from the classical dynamical twisting, the condition for the existence of the 'good' Lax representation is found. In section 3, after reviewing the quantum Z_n-symmetric Belavin model, we construct the classical Z_n-symmetric r-matrix. After reviewing Sklyanin's work on the elliptic CM model in section 4, in section 5 we construct the 'good' Lax representation for the elliptic A_{n-1} CM model which possesses a non-dynamical r-matrix structure. Finally, we give a summary and discussion in section 6. The proof of the main result (proposition 4) is given in an appendix.

2. The dynamical twisting of the classical r-matrix

In this paper, we only deal with completely integrable finite-particle systems. In this section we will review some general theories of completely integrable finite-particle systems

A Lax pair (L, M) consists of two functions on the phase space of the system with values in some Lie algebra g, such that the evolution equations may be written in the following form:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M] \tag{1}$$

where [,] denotes the bracket in the Lie algebra g. Our interest in the existence of such a pair lies in the fact that it allows for an easy construction of conserved quantities (integrals of motion)—it follows that the adjoint-invariant quantities $\operatorname{tr} L^n$ are the integrals of motion. In order to apply the Liouville theorem to this set of possible action variables we need them to be Poisson-commuting. As shown in [7], for the commutativity of the integrals $\operatorname{tr} L^n$ of the Lax operator it is neccessary and sufficient that the fundamental Poisson bracket $\{L_1(u), L_2(v)\}$

can be represented in the commutator form

$$\{L_1(u), L_2(v)\} = [r_{12}(u, v), L_1(u)] - [r_{21}(v, u), L_2(v)] \tag{2}$$

where we use the notation

$$L_1 \equiv L \otimes 1 \qquad L_2 \equiv 1 \otimes L \qquad r_{21} = P r_{12} P$$

and P is the permutation operator such that $P x \otimes y = y \otimes x$.

Generally speaking, the r-matrix $r_{12}(u, v)$ does depend on the dynamical variables. For some special cases where $r_{12}(u, v)$ is independent of the dynamical variables, the r-matrix is called the non-dynamical r-matrix, and has been studied extensively [15]. In contrast to the extensively studied case of the non-dynamical r-matrix, no general theory of the dynamical r-matrix exists at the moment, apart from a few concrete examples and observations. Still, the collection of examples is rather sparse, and any new example of dynamical r-matrix could possibly contribute to better understanding of their algebraic and geometric nature.

The Poisson bracket structure (2) obeys a Jacobi identity which implies an algebraic constraint for the r-matrix. Since the r-matrix may depend on the dynamical variables this constraint takes a complicated form:

$$\big[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}\big] + \text{cycl. perm} = 0. \tag{3}$$

Using relevant particular cases of this general identity, one can obtain the classical Yang–Baxter equation for the non-dynamical r-matrix and the classical dynamical Yang–Baxter equation [3, 7] for the dynamical one. It should be remarked that such a classification is by no means unique, which drastically depend on the Lax representation that one choose for a system. Namely, there is no one-to-one correspondence between a given dynamical system and a defined r-matrix, a same dynamical system may have several Lax representations and several r-matrix . The different Lax representations of a system are conjugated with each other: if $(\widetilde{L}, \widetilde{M})$ is one of other Lax pair of the same dynamical system conjugated with the old one (L, M), it means that

$$\begin{aligned} &\frac{\mathrm{d}\widetilde{L}}{\mathrm{d}t} = [\widetilde{L}, \widetilde{M}] \\ &\widetilde{L}(u) = g(u)L(u)g^{-1}(u) \qquad \widetilde{M}(u) = g(u)M(u)g^{-1}(u) - \left(\frac{\mathrm{d}}{\mathrm{d}t} g(u)\right) g^{-1}(u) \end{aligned} \tag{4}$$

where $g(u) \in G$ whose Lie algebra is g. Then, we have

Proposition 1. *The Lax pair $(\widetilde{L}, \widetilde{M})$ has the following r-matrix structure:*

$$\{\widetilde{L}_1(u), \widetilde{L}_2(v)\} = [\widetilde{r}_{12}(u, v), \widetilde{L}_1(u)] - [\widetilde{r}_{21}(v, u), \widetilde{L}_2(v)] \tag{4a}$$

where

$$\begin{aligned} \widetilde{r}_{12}(u, v) = {} & g_1(u)g_2(v)r_{12}(u, v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u), L_2(v)\}g_1^{-1}(u)g_2^{-1}(v) \\ & + \tfrac{1}{2}\big[\{g_1(u), g_2(v)\}g_1^{-1}(u)g_2^{-1}(v)\ ,\ g_2(v)L_2(v)g_2^{-1}(v)\big]. \end{aligned} \tag{5}$$

Proof. The proof is direct by substitution of (4) and (4a) in the fundamental Poisson bracket and use of the following identity:

$$\big[[s_{12}, L_1], L_2\big] = \big[[s_{12}, L_2], L_1\big]$$

where s_{12} is any matrix on $g \otimes g$. □

It can be seen that: (i) the Lax operator L is transformed under a similarity transformation from the different Lax representation (only for finite-particle systems); (ii) The corresponding M has undergone the usual gauge transformation; (iii) The r-matrix is transformed as some generalized gauge transformation, which can be considered as the classical version of the dynamically twisting relation between the quantum R-matrix [4]. Therefore, it is of great value to find a 'good' Lax representation for a system if it exists, in which the corresponding r-matrix is non-dynamical one and the extensively studied theories [4, 15] can be directly applied to the system—such as the dressing transformation, quantization, etc.

Corollary to proposition 1. *For given Lax pair (L, M) and the corresponding r-matrix, if there exists a g such that*

$$h_{12} = \Big\{g_1(u)g_2(v)r_{12}(u,v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u), L_2(v)\}g_1^{-1}(u)g_2^{-1}(v)$$
$$+ \ \tfrac{1}{2}\Big[\{g_1(u), g_2(v)\}g_1^{-1}(u)g_2^{-1}(v)\ ,\ g_2(v)L_2(v)g_2^{-1}(v)\Big]\Big\}$$
$$\textit{and} \qquad \partial_{q_i} h_{12} = \partial_{p_j} h_{12} = 0 \tag{6}$$

the non-dynamical Lax representation of the system exists.

By straightforward calculation, we also have

Proposition 2. *The twisting Lax pair $(\widetilde{L}, \widetilde{M})$ and the corresponding r-matrix $\widetilde{r}_{12}$ satisfies*
$[\widetilde{L}_1, [\widetilde{r}_{12}, \widetilde{r}_{13}] + [\widetilde{r}_{12}, \widetilde{r}_{23}] + [\widetilde{r}_{32}, \widetilde{r}_{13}] + \{\widetilde{L}_2, \widetilde{r}_{13}\} - \{\widetilde{L}_3, \widetilde{r}_{12}\}] + \textit{cycl. perm} = 0.$

The main purpose of this paper is to find a 'good' Lax representation for the elliptic A_{n-1} CM model.

3. The elliptic function and the elliptic Z_n-symmetric R- and r-matrices

We first briefly review the elliptic Z_n-symmetric quantum R-matrix which is related to the Z_n-symmetric Belavin model [8, 19, 21, 25]. For $n \in Z_+, n \geqslant 2$, we define the $n \times n$ matrices h, g, I_α as

$$h_{ij} = \delta_{i+1,j \mod n} \qquad g_{ij} = \omega^i \delta_{i,j} \qquad I_{\alpha_1,\alpha_2} \equiv I_\alpha = g^{\alpha_2} h^{\alpha_1}$$

where $\alpha_1, \alpha_2 \in Z_n$ and $\omega = \exp(2\pi \frac{\sqrt{-1}}{n})$. We also define some elliptic functions

$$\theta^{(j)}(u) = \theta\begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{bmatrix}(u, n\tau) \qquad \sigma(u) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u, \tau) \tag{7}$$

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau) = \sum_{m=-\infty}^{\infty} \exp\Big\{\sqrt{-1}\pi\big[(m+a)^2\tau + 2(m+a)(z+b)\big]\Big\}$$

$$\theta'^{(j)}(u) = \partial_u\{\theta^{(j)}(u)\} \qquad \sigma'(u) = \partial_u\{\sigma(u)\} \qquad \xi(u) = \partial_u\Big\{\ln\sigma(u)\Big\} \tag{8}$$

$$E(u,v) = \frac{\sigma(u+v)}{\sigma(u)\sigma(v)} \tag{9}$$

where τ is a complex number with $\mathrm{Im}\,(\tau) > 0$. Then we define the Z_n-symmetric Belavin R-matrix [19] as

$$R_{ij}^{lk}(v) = \begin{cases} \dfrac{\theta'^{(0)}(0)\sigma(v)\sigma(w)}{\sigma'(0)\theta^{(0)}(v)\sigma(v+w)} \dfrac{\theta^{(0)}(v)\theta^{(i-j)}(v+w)}{\theta^{(i-l)}(w)\theta^{(l-j)}(v)} & \text{if } i+j = l+k \mod n \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where w is a complex number which is called the crossing parameter of the R-matrix. We should remark that our R-matrix coincides with the usual one [19, 21] up to a scalar factor

$$\frac{\theta'^{(0)}(0)\sigma(v)}{\sigma'(0)\theta^{(0)}(v)} \prod_{j=1}^{n-1} \frac{\theta^{(j)}(v)}{\theta^{(j)}(0)}$$

which is required to satisfy equation (15). The R-matrix satisfies the quantum Yang–Baxter equation (QYBE)

$$R_{12}(v_1 - v_2)R_{13}(v_1 - v_3)R_{23}(v_2 - v_3) = R_{23}(v_2 - v_3)R_{13}(v_1 - v_3)R_{12}(v_1 - v_2). \tag{11}$$

Moreover, the R-matrix possesses in following $(Z_n \otimes Z_n)$-symmetric properties

$$R_{12}(v) = (a \otimes a)R_{12}(v)(a \otimes a)^{-1} \qquad \text{for} \quad a = g, h. \tag{12}$$

We introduce an $n \otimes n$ matrix $\widehat{T}(v)$, where the matrix elements $\widehat{T}(v)_i^j$ are operators, which satisfies the equation (also called a QYBE)

$$R_{12}(v_1 - v_2)\widehat{T}_1(v_1)\widehat{T}_2(v_2) = \widehat{T}_2(v_2)\widehat{T}_1(v_1)R_{12}(v_1 - v_2). \tag{13}$$

Next we turn to the factorized difference representation for the operator $\widehat{T}(v)$ [17, 19, 25].

Set an $n \otimes n$ matrix $A(u; q)$

$$A(u; q)^i_j \equiv A(u; q_1, q_2, \ldots, q_n)^i_j = \theta^{(i)}\left(u + nq_j - \sum_{k=1}^n q_k + \frac{n-1}{2}\right) \tag{14}$$

where $A(u, q)^i_j$ corresponds to the intertwiner function $\varphi^{(i)}_j$ between the Z_n-symmetric Belavin R-matrix and the $A^{(1)}_{n-1}$ face model [21] in [19]. Construct the operator $\widehat{T}(u)$

$$\widehat{T}(u)^i_j = A(u + sw; q)^i_k A^{-1}(u; q)^k_j D_k \tag{14a}$$

where s is a complex number associated with the representation of the Sklyanin algebra [19] and which will be related to the coupling constant of the elliptic A_{n-1} CM model, equation (A3), and D_k is a difference operator such that

$$D_k f(q) \equiv D_k f(q_1, q_2, \ldots, q_n) = f(q_1, \ldots, q_{k-1}, q_k - w, q_{k+1}, \ldots, q_n).$$

Then following the results in [16], we have

Theorem 1 ([16, 29, 30]). *The L-operator $\widehat{T}(u)$ defined in (14a) satsifes the QYBE (13).*

We can define a corresponding Z_n-symmetric (classical) r-matrix which has the following relationship with the R-matrix:

$$\begin{aligned} &R_{12}(v)|_{w=0} = 1 \otimes 1 \\ &R_{12}(v) = 1 \otimes 1 + w r_{12}(v) + 0(w^2) \qquad \text{when the crossing parameter } w \longrightarrow 0. \end{aligned} \tag{15}$$

Then we have

Proposition 3. *The corresponding elliptic Z_n-symmetric r-matrix is*

$$r^{lk}_{ij}(v) = \begin{cases} (1 - \delta^l_i)\dfrac{\theta'^{(0)}(0)\theta^{(i-j)}(v)}{\theta^{(l-j)}(v)\theta^{(i-l)}(0)} \\ \qquad\qquad + \delta^l_i\delta^k_j\left(\dfrac{\theta'^{(i-j)}(v)}{\theta^{(i-j)}(v)} - \dfrac{\sigma'(v)}{\sigma(v)}\right) & \textit{if } i + j = l + k \mod n \\ 0 & \textit{otherwise} \end{cases} \tag{16}$$

and it satisfies the non-dynamical (classical) Yang–Baxter equation and antisymmetric properties

$$\begin{aligned} &[r_{12}(v_1 - v_2), r_{13}(v_1 - v_3)] + [r_{12}(v_1 - v_2), r_{23}(v_2 - v_3)] \\ &\qquad + [r_{13}(v_1 - v_3), r_{23}(v_2 - v_3)] = 0 \end{aligned} \tag{17}$$

$$-r_{21}(-v) = r_{12}(v). \tag{18}$$

Proof. When $w \longrightarrow 0$, we have the following asympotic properties:

$$\sigma(w) = w\sigma'(0) + 0(w^3) \qquad \theta^{(0)}(w) = w\theta^{'(0)}(0) + 0(w^3)$$

$$\theta^{(i)}(w) = \theta^{(i)}(0) + w\theta^{'(i)}(0) + 0(w^2) \qquad i \neq 0 \mod n.$$

Then, when $w \longrightarrow 0$, we have

$$\frac{\theta^{'(0)}(0)\sigma(v)}{\sigma'(0)\theta^{(0)}(v)} \prod_{m=1}^{n-1} \frac{\theta^{(m)}(v)}{\theta^{(m)}(0)} R_{ij}^{lk}(v)$$

$$= w(1-\delta_i^l)\frac{\sigma'(o)}{\sigma(v)} \prod_{m=1}^{n-1} \frac{\theta^{(m)}(v)}{\theta^{(m)}(0)} \frac{\theta^{(0)}(v)\theta^{(i-j)}(v)}{\theta^{(l-j)}(v)\theta^{(i-l)}(v)} + 0(w^2)$$

$$+ \delta_i^l \frac{\sigma'(0) + 0(w^2)}{\sigma(v) + w\sigma'(v) + 0(w^2)}$$

$$\times \prod_{m=1}^{n-1} \frac{\theta^{(m)}(v)}{\theta^{(m)}(0)} \frac{\theta^{(0)}(v)(\theta^{(i-j)}(v) + w\theta'^{(i-j)}(v) + 0(w^2)}{\theta^{(i-j)}(v)(\theta'^{(0)}(0) + 0(w^2))}$$

$$= \prod_{m=1}^{n-1} \frac{\theta^{(m)}(v)}{\theta^{(m)}(0)} \frac{\sigma'(0)\theta^{(0)}(v)}{\theta'^{(0)}(0)\sigma(v)} \delta_i^l \delta_j^k + w \prod_{m=1}^{n-1} \frac{\theta^{(m)}(v)}{\theta^{(m)}(0)} \left\{ (1-\delta_i^l) \frac{\sigma'(0)\theta^{(0)}(v)\theta^{(i-j)}(v)}{\sigma(v)\theta^{(l-j)}(v)\theta^{(i-l)}(0)} \right.$$

$$\left. + \delta_i^l \delta_j^k \frac{\sigma'(0)\theta^{(0)}(v)}{\sigma(v)\theta'^{(0)}(0)} \left(\frac{\theta'^{(i-j)}(v)}{\theta^{(i-j)}(v)} - \frac{\sigma'(v)}{\sigma(v)} \right) \right\} + 0(w^2).$$

By the definition of the classical r-matrix from the quantum one, equation (15), we have (16). The classical Yang–Baxter equation (17) is the direct results of the QYBE and the asympotic properties (18). The antisymmetric properties of the r-matrix can be derived from the following relations between the θ-functions:

$$\theta^{(\alpha)}(v) = -\mathrm{e}^{2\sqrt{-1}\pi\alpha}\,\theta^{(-\alpha)}(-v) \qquad \frac{\theta'^{(\alpha)}(v)}{\theta^{(\alpha)}(v)} = -\frac{\theta'^{(-\alpha)}(-v)}{\theta^{(-\alpha)}(-v)}.$$

□

One can also check that the classical r-matrix $r_{12}(u)$ possesses the following $(Z_n \otimes Z_n)$-symmetric properties:

$$r_{12}(v) = (a \otimes a) r_{12}(v) (a \otimes a)^{-1} \qquad \text{for } a = g, h.$$

4. Review of the elliptic A_{n-1} CM model

The elliptic A_{n-1} CM model is a system of n one-dimensional particles interacting according to the two-body potential

$$V(q_{ij}) = \gamma Q(q_{ij}) \qquad q_{ij} = q_i - q_j \qquad i, j = 1, \ldots, n \tag{19}$$

$$Q(v) - Q(u) = E(u, v)E(u, -v) \tag{20}$$

where γ is the coupling constant, $Q(u)$ is a Weierstrass function and the elliptic function $E(u, v)$ is defined in (9). In terms of the canonical variables $\{p_i, q_j\}$ $(i, j = 1, \ldots, n)$ with the canonical Poisson bracket

$$\{p_i, p_j\} = \{q_l, q_k\} = 0 \qquad \{p_i, q_j\} = \delta_{ij} \qquad i, j, k, l = 1, \ldots, n \tag{21}$$

the Hamiltonian of the system is expressed as

$$H = \sum_{i=1}^{n} p_i^2 + \sum_{i \neq j} V(q_{ij}). \tag{22}$$

The above Hamiltonian with the potential (19) is known to be completely integrable [13, 22–24]. The most effective way to show its integrability is to construct the Lax representation for the system. One Lax pair (L, M) was first found by Krichever [22]. The Krichever Lax operator (or L-operator) is

$$L^i_j(u) = p_i \delta^i_j + (1 - \delta^i_j)\sqrt{\gamma} E(u, q_{ji}) \tag{23}$$

where u is spectra parameter and the motion equation can be rewritten in the Lax form

$$\frac{\mathrm{d}}{\mathrm{d}t} L(u) = \{L(u), H\} = [L(u), M(u)].$$

The Hamiltionian defined in (22) can be rewritten in terms of the Poisson-commuting family $\{\mathrm{tr}\, L^l(u)\}$ $(l = 1, \ldots, n)$, which forms the enough independent integrals

$$H = \mathrm{tr}(L^2(u)) + V(u). \tag{24}$$

$V(u)$ does not depend upon the dynamical variables and the identity (20) is used. The r-matrix structure of this Lax operator was given by Sklyanin [30] and Braden *et al* [10]. The fundamental Poisson bracket of the Lax operator can be described in the r-matrix form [30]

$$\{L_1(u), L_2(v)\} = [r_{12}(u, v), L_1(u)] - [r_{21}(v, u), L_2(v)] \tag{25}$$

and the dynamical r-matrix $r_{12}(u, v)$ is

$$r_{12}(u, v) = a \sum_{i=1}^{n} E^{ii}_{ii} + \sum_{i \neq j} c_{ij} E^{ij}_{ji} + \sum_{i \neq j} d_{ij} (E^{ii}_{ij} + E^{ji}_{jj}) \tag{26}$$

where

$$E^{lk}_{ij} = E^l_i \otimes E^k_j \qquad a = r^{ii}_{ii} = -\xi(u - v) - \xi(v) \tag{27}$$

$$c_{ij} = r^{ij}_{ji} = \sqrt{-\gamma} E(u - v, q_{ij}) \qquad d_{ij} = r^{ii}_{ij} = r^{ji}_{jj} = \tfrac{1}{2}\sqrt{-\gamma} E(v, q_{ij}) \tag{28}$$

where the elliptic function $\xi(u)$ is defined in (8). Sklyanin also shown that the dynamical r-matrix $r_{12}(u, v)$ defined in (26) satisfies the dynamical Yang–Baxter (or generalized Yang–Baxter) equation [30]

$$[R^{(123)}, L_1] + [R^{(231)}, L_2] + [R^{(321)}, L_3] = 0 \tag{29}$$

where

$$R^{(123)} \equiv r_{(123)} - \{r_{13}, L_2\} + \{r_{12}, L_3\}$$

and

$$r_{(123)} \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] - [r_{13}, r_{32}].$$

The Jacobi identity of the fundamental Poisson bracket is the results of (29). Due to the dynamical properties of the r-matrix $r_{12}(u, v)$, the Poisson bracket of L-operator is no longer closed. The quantum version of (29) and the generalized (dynamical) Yang–Baxter equation has still not been found, except the spin generalization of the CM model for which the Gervais–Neveu–Felder equation has been found [4, 30].

5. The 'good' Lax representaion of the elliptic A_{n-1} CM model and its r-matrix

The L-operator of the elliptic A_{n-1} CM model given by Krichever in (23) and the corresponding r-matrix $r_{12}(u, v)$ given by Sklyanin in (26) lead to some difficulties [30] in the investigation of the CM model. This motivates us to find a 'good' Lax representation of the CM model. As can be seen from proposition 1 and its corollary in section 3, this means finding a $g(u)$ in (4) which satisfies (6). Fortunately, we can find such a $g(u)$, from which we construct a new L-operator $\widetilde{L}(u)$ of the elliptic A_{n-1} CM model (this kind of L-operator does not always exist for a general completely integrable system). The corresponding r-matrix of $\widetilde{L}(u)$ is purely numerical, and is equal to the classical Z_n-symmetric r-matrix. For comparison with the L-operator given by Krichever, we call this L-operator found by us the new Lax operator.

Define

$$\begin{aligned} &g(u) = A(u;q)\Lambda(q) \qquad \Lambda(q)^i_j = h_i(q)\delta^i_j \\ &h_j(q) \equiv h_j(q_1,\ldots,q_n) = \frac{1}{\prod_{l\neq i}\sigma(q_{il})} \end{aligned} \tag{30}$$

where $A(u;q)^i_j$ is defined in (14). Let us construct the new L-operator $\widetilde{L}(u)$:

$$\begin{aligned} &\widetilde{L}(u) = g(u)L(u)g^{-1}(u) \\ &\widetilde{M}(u) = g(u)M(u)g^{-1}(u) - \left(\frac{\mathrm{d}}{\mathrm{d}t}g(u)\right)g^{-1}(u). \end{aligned} \tag{31}$$

Then we have

Proposition 4. *The fundamental Poisson bracket of the L-operator $\widetilde{L}(u)$ can be written in the usual Poisson–Lie form with a purely numerical r-matrix*

$$\{\widetilde{L}_1(u), \widetilde{L}_2(v)\} = [\widetilde{r}_{12}(u-v), \widetilde{L}_1(u) + \widetilde{L}_2(v)] \tag{32}$$

and the corresponding r-matrix $\widetilde{r}_{12}(u)$ is a non-dynamical one, namely the Z_n-symmetric r-matrix defined in (16).

Proof. The proof is given in the appendix.

The most important property of this new Lax operator is that the corresponding r-matrix does not depend upon the dynamical variables. Consequently, the extensively studied theory for the non-dynamical system [15] can be used to study the elliptic A_{n-1} CM model.

The Z_n-symmetric r-matrix $\widetilde{r}_{12}(u)$ can also be obtained from the classical dynamical twisting as follows:

$$\begin{aligned} \widetilde{r}_{12}(u, v) = {} & g_1(u)g_2(v)r_{12}(u, v)g_1^{-1}(u)g_2^{-1}(v) \\ & + g_2(v)\{g_1(u), L_2(v)\}g_1^{-1}(u)g_2^{-1}(v) \end{aligned} \tag{33}$$

up to some matrix which commutes with $L_1(u) + L_2(v)$.

The standard Poisson–Lie bracket (32) of L-operator $\widetilde{L}(u)$ and the numerical r-matrix $\widetilde{r}_{12}(u)$ satisfying the classical Yang–Baxter equation (17) and antisymmetry (18) make it possible to construct the quantum theory of the elliptic A_{n-1} CM model. Moreover, the numerical r-matrix $\widetilde{r}_{12}(u)$ could provide a means of constructing a separation of variables for the elliptic A_{n-1} CM model in the same manner as in the case of the integrable magnetic chain [30]. It also makes it possible to construct the dressing transformation for the model. The dressing group of this system would be analogous of the semi-classical limit of the Z_n Sklyanin algebra.

6. Discussion

In this paper, we have constructed the non-dynamical r-matrix structure just for the elliptic A_{n-1} Calogero–Moser model. Such a 'good' Lax representation for the degenerate case, the rational, trigonometric and hyperbolic CM model, could also could be constructed. It would also be very interesting to construct such a classical dynamical twisting for the Ruijsenaars–Schneider model.

Appendix. Proof of proposition 4

In this appendix we give the proof the proposition 4, which is the main result of our paper.

Set the classical L-operator $T(u)$ as follows:

$$T(u)^i_j = \sum_k A(u;q)^i_k A^{-1}(u;q)^k_j p_k - s(\partial_u A(u;q))^i_k A^{-1}(u;q)^k_j$$

where $\{p_k\}$ is the classical moment which is conjugated with $\{q_k\}$, and $\{p_i, q_j\}$ satisfies the canonical Poisson bracket (21).

Lemma A1. *The classical operator $T(u)$ has the standard Poisson–Lie bracket*

$$\{T_1(u), T_2(v)\} = [\widetilde{r}_{12}(u-v), T_1(u) + T_2(v)] \tag{A1}$$

where the r-matrix $\widetilde{r}_{12}(u)$ is the Z_n-symmetric r-matrix defined in (16).

Proof. When $w \longrightarrow 0$, the quantum difference L-operator $\widehat{T}(u)$ has the following asympotic properties:

$$\begin{aligned}
\widehat{T}(u)^i_j &= \sum_k A(u;q)^i_k A^{-1}(u;q)^k_j - w \sum_k A(u;q)^i_k A^{-1}(u;q)^k_j \frac{\partial}{\partial q_k} \\
&\qquad + sw \sum_k (\partial_u A(u;q))^i_k A^{-1}(u;q)^k_j + 0(w^2) \\
&\equiv \delta^i_j - w\widehat{T}^{(1)}(u)^i_j + 0(w^2)
\end{aligned}$$

where

$$\widehat{T}^{(1)}(u)^i_j = \sum_k A(u;q)^i_k A^{-1}(u;q)^k_j \frac{\partial}{\partial q_k} - s\sum_k (\partial_u A(u;q))^i_k A^{-1}(u;q)^k_j.$$

From the QYBE (13) we have

$$[\widehat{T}^{(1)}_1(u), \widehat{T}^{(1)}_2(v)] = [\widetilde{r}_{12}(u-v), \widehat{T}^{(1)}_1(u) + \widehat{T}^{(1)}_2(v)].$$

If we use p_k instead of the differential $\frac{\partial}{\partial q_k}$ and the classical L-operator $T(u)$ instead of $\widehat{T}^{(1)}(u)$, we have equation (A1). □

Lemma A2. *The $T(u)^i_j$ can be written explictly as follows:*

$$T(u)^i_j = \sum_{k,k'} \widetilde{g}(u)^i_k \overline{T}(u)^k_{k'} \widetilde{g}^{-1}(u)^{k'}_j \equiv \left\{(A(u;q)\Lambda(q))\overline{T}(u)\big(A(u;q)\Lambda(q)\big)^{-1}\right\}^i_j \tag{A2}$$

where $\overline{T}(u)$ and $\Lambda(q)$ are

$$\overline{T}(u)^i_j = \left(p_i - \frac{\partial}{\partial q_k}\ln \Delta^{s/n}(q)\right)\delta^i_j + \sqrt{\gamma}(1-\delta^i_j)E(u;q_{ji}) \tag{A3}$$

$$\Delta(q) = \prod_{i<j}\sigma(q_{ij}) \qquad \textit{coupling constant}\ \ \gamma = \left(-\frac{s\sigma'(0)}{n}\right)^2 \qquad \Lambda(q)^i_j = \frac{1}{\prod_{l\neq i}\sigma(q_{il})}\delta^i_j.$$

Proof. In order to calculate the matrix element $(\partial_u(A(u;q))A^{-1}(u;q)$, we first consider

$$A(u+w;q)A^{-1}(u;q) = A(u;q)[A^{-1}(u;q)A(u+w;q)]A^{-1}(u;q).$$

From the defintion of $A(u;q)^i_j$, equation (14), and the Vandermonde-type determinat formula

$$\det[\theta^{(j)}(u_k)] = \text{constant} \times \sigma\left(\frac{1}{n}\sum_k u_k - \frac{n-1}{2}\right)\prod_{1\leqslant j<k\leqslant n}\sigma\left(\frac{u_k-u_j}{n}\right)$$

where the constant does not depend upon $\{u_k\}$, we have

$$\begin{aligned}\left[A^{-1}(u;q)A(u+w;q)\right]^i_j &= \sum_k A^{-1}(u;q)^i_k A(u+w;q)^k_j\\ &= \frac{\sigma(w/n+u+q_{ji})}{\sigma(u)}\prod_{k\neq i}\frac{\sigma(w/n+q_{jk})}{\sigma(q_{ik})}.\end{aligned}$$

We then have

$$\begin{aligned}(A^{-1}(u;q)\partial_u A(u;q))^i_j &= \frac{\partial}{\partial w}\left\{\frac{\sigma(w/n+u+q_{ji})}{\sigma(u)}\prod_{k\neq i}\frac{\sigma(w/n+q_{jk})}{\sigma(q_{ik})}\right\}\Bigg|_{w=0}\\
&= \frac{1}{n}\left\{\frac{\sigma'(u)}{\sigma(u)}\delta^i_j + \frac{\sigma(u+q_{ji})}{\sigma(u)}\left(\delta^i_j\sum_{k\neq i}\frac{\sigma'(q_{ik})}{\sigma(q_{ik})} + (1-\delta^i_j)\sigma'(0)\frac{\prod_{k\neq i,j}\sigma(q_{jk})}{\prod_{k\neq i}\sigma(q_{ik})}\right)\right\}\\
&= \frac{1}{n}\left\{\left(\frac{\sigma'(u)}{\sigma(u)} + \sum_{k\neq i}\frac{\sigma'(q_{ik})}{\sigma(q_{ik})}\right)\delta^i_j + (1-\delta^i_j)\frac{\sigma'(0)\sigma(u+q_{ji})}{\sigma(u)\sigma(q_{ji})}\frac{\prod_{k\neq j}\sigma(q_{jk})}{\prod_{k\neq i}\sigma(q_{ik})}\right\}\\
&= \frac{1}{n}\left\{\left(\frac{\sigma'(u)}{\sigma(u)} + \frac{\partial}{\partial q_j}\big(\ln\Delta(q)\big)\right)\delta^i_j - (1-\delta^i_j)(-\sigma'(0))E(u;q_{ji})\frac{\prod_{k\neq j}\sigma(q_{jk})}{\prod_{k\neq i}\sigma(q_{ik})}\right\}\\
&= \frac{1}{\prod_{k\neq i}\sigma(q_{ik})}\left\{\left(\frac{\sigma'(u)}{n\sigma(u)} + \frac{\partial}{\partial q_j}\big(\ln\Delta^{1/n}(q)\big)\right)\delta^i_j\right.\\
&\quad\left. -(1-\delta^i_j)\left(-\frac{\sigma'(0)}{n}\right)E(u;q_{ji})\right\}\prod_{k\neq i}\sigma(q_{ik}).\end{aligned}$$

Substituting $A^{-1}(u;q)\partial_u A(u;q)$ in the defintion of $T(u)$, we have

$$\begin{aligned}T(u)^i_j &= \sum_k A(u;q)^i_k p_k A^{-1}(u;q)^k_j - s\sum_{k,k'}A(u;q)^i_k(A^{-1}(u;q)\partial_u A(u;q))^k_{k'}A^{-1}(u;q)^{k'}_j\\
&= (A(u;q)\Lambda(q))^i_k\left\{\left(p_k - \frac{s\sigma'(u)}{n\sigma(u)} + \frac{\partial}{\partial q_k}\big(\ln\Delta^{s/n}(q)\big)\right)\delta^k_{k'}\right.\\
&\quad\left. + (1-\delta^k_{k'})\sqrt{\gamma}E(u;q_{ji})\right\}(\Lambda(q)^{-1})A(u;q)^{-1})^{k'}_j.\end{aligned}$$

□

We consider a map

$$\begin{cases} p_i \longrightarrow p_i - \dfrac{\partial}{\partial q_i}\big(\ln\Delta^{s/n}(q)\big)\\ q_i \longrightarrow q_i.\end{cases} \tag{A4}$$

Lemma A3. *The map defined in (A4) is a Poisson map [1] (or a canonical transformation).*

Proof. Lemma A3 can be proved by considering the symplectic two-form

$$\sum_i \mathrm{d}\left(p_i - \frac{\partial}{\partial q_i}\ln \Delta^{s/n}(q)\right) \wedge \mathrm{d}q_i$$
$$= \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i + \sum_{ij}\left(\frac{\partial^2}{\partial q_i \partial q_j}\ln \Delta^{s/n}(q)\right)\mathrm{d}q_i \wedge \mathrm{d}q_j$$
$$= \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i .$$

□

Since the Poisson bracket is invariant under the Poisson map [1], we have proposition 4 from lemmas A1 and A3.

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